

# MAXIMAL COHEN–MACAULAY APPROXIMATIONS AND SERRE’S CONDITION

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*Dedicated to Professor Ngo Viet Trung on the occasion of his sixtieth birthday*

**ABSTRACT.** This paper studies the relationship between Serre’s condition  $(R_n)$  and Auslander–Buchweitz’s maximal Cohen–Macaulay approximations. It is proved that a Gorenstein local ring satisfies  $(R_n)$  if and only if every maximal Cohen–Macaulay module is a direct summand of a maximal Cohen–Macaulay approximation of a (Cohen–Macaulay) module of codimension  $n + 1$ .

## 1. INTRODUCTION

In the 1980s, Auslander and Buchweitz [2] introduced the notion of a maximal Cohen–Macaulay approximation of a finitely generated module over a Cohen–Macaulay local ring with a canonical module, which has been playing a fundamental role in the representation theory of Cohen–Macaulay rings. Several years ago Kato [6] gave the following characterization theorem of Gorenstein local rings by maximal Cohen–Macaulay approximations. We abbreviate Cohen–Macaulay to CM and maximal Cohen–Macaulay to MCM.

**Theorem 1.1** (Kato). *Let  $R$  be a  $d$ -dimensional Gorenstein local ring.*

- (1) *The following are equivalent for  $d \geq 1$ .*
  - (a)  *$R$  is a domain.*
  - (b) *Every MCM  $R$ -module is a MCM approximation of a (CM)  $R$ -module of codimension 1.*
- (2) *The following are equivalent for  $d \geq 2$ .*
  - (a)  *$R$  is a unique factorization domain.*
  - (b) *Every MCM  $R$ -module is a MCM approximation of a (CM)  $R$ -module of codimension 2.*

It is natural to ask what happens if in the statements (b) of the above theorem we weaken the condition of being a MCM approximation to that of being a direct summand of a MCM approximation. The main purpose of this paper is to answer this question in more general settings. Our main results yield the following theorem.

**Theorem 1.2.** *Let  $R$  be a  $d$ -dimensional Gorenstein local ring. The following are equivalent for each  $0 \leq c \leq d$ .*

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- (1)  $R$  satisfies Serre's condition  $(R_{c-1})$ .
- (2) Every MCM  $R$ -module is a direct summand of a  $c$ -th syzygy of a (CM)  $R$ -module of codimension  $c$ .
- (3) Every MCM  $R$ -module is a direct summand of a MCM approximation of a (CM)  $R$ -module of codimension  $c$ .

Letting  $c = 1, 2$  in the above theorem, we obtain the following result which is analogous to Kato's theorem. This gives the answer to the question raised above.

**Corollary 1.3.** *Let  $R$  be a  $d$ -dimensional Gorenstein local ring.*

- (1) *The following are equivalent for  $d \geq 1$ .*
  - (a)  $R$  is reduced.
  - (b) Every MCM  $R$ -module is a direct summand of a MCM approximation of a (CM)  $R$ -module of codimension 1.
- (2) *The following are equivalent for  $d \geq 2$ .*
  - (a)  $R$  is normal.
  - (b) Every MCM  $R$ -module is a direct summand of a MCM approximation of a (CM)  $R$ -module of codimension 2.

This paper is organized as follows. In Section 2, we consider over a CM local ring the condition that all MCM modules are direct summands of syzygies of certain modules. In Section 3, we study over a Gorenstein local ring the condition that all MCM modules are direct summands of MCM approximations of certain modules. The proof of Theorem 1.2 is given at the end of this section.

## 2. MCM MODULES THAT ARE DIRECT SUMMANDS OF SYZYGIES

Throughout this paper, let  $R$  be a commutative Cohen–Macaulay local ring of Krull dimension  $d$ . All  $R$ -modules are assumed to be finitely generated.

Let us begin with recalling some basic definitions.

**Definition 2.1.** (1) For an integer  $n \geq 0$  we denote by  $\Omega^n M$  an  $n$ -th syzygy of  $M$ , that is, the image of the  $n$ -th differential map in a free resolution of  $M$ .  
(2) For an integer  $n \geq -1$  we say that  $R$  satisfies *Serre's condition*  $(R_n)$  if the local ring  $R_{\mathfrak{p}}$  is regular for all prime ideals  $\mathfrak{p}$  of  $R$  with  $\text{ht } \mathfrak{p} \leq n$ .  
(3) The *singular locus*  $\text{Sing } R$  of  $R$  is by definition the set of prime ideals  $\mathfrak{p}$  of  $R$  such that the local ring  $R_{\mathfrak{p}}$  is nonregular.  
(4) Let  $M$  be an  $R$ -module. The *nonfree locus*  $\text{NF}(M)$  (respectively, the *infinite projective dimension locus*  $\text{IPD}(M)$ ) of  $M$  is defined as the set of prime ideals  $\mathfrak{p}$  of  $R$  such that the  $R_{\mathfrak{p}}$ -module  $M_{\mathfrak{p}}$  is nonfree (respectively, is of infinite projective dimension).  
(5) Let  $V$  be a closed subset of  $\text{Spec } R$ . Then we set  $\text{codim } V = d - \dim V$  and call this the *codimension* of  $V$ . The *codimension*  $\text{codim } M$  of an  $R$ -module  $M$  is defined as the codimension of  $\text{Supp } M$ , whence  $\text{codim } M = d - \dim M$ .

**Remark 2.2.** (1) If  $X, Y$  are  $n$ -th syzygies of an  $R$ -module  $M$ , then  $X \oplus F \cong Y \oplus G$  for some free  $R$ -modules  $F, G$ .  
(2) By definition  $R$  always satisfies  $(R_{-1})$ .

- (3) It is well-known and easy to see that the nonfree locus and the infinite projective dimension locus of an  $R$ -module are always closed subsets of  $\operatorname{Spec} R$  in the Zariski topology.  
(4) If  $M$  is a MCM  $R$ -module, then  $\operatorname{NF}(M)$  is contained in  $\operatorname{Sing} R$ .

In the following proposition we study how to represent each MCM module as a direct summand of a syzygy of a certain CM module. This result will become a basis of our main results.

**Proposition 2.3.** *Let  $M$  be a MCM  $R$ -module. Then for each integer  $0 \leq c \leq \operatorname{codim} \operatorname{NF}(M)$  there exists a CM  $R$ -module  $N$  such that*

- (1)  $\operatorname{codim} N = c$ ,
- (2)  $\operatorname{IPD}(N) = \operatorname{NF}(M)$  and
- (3)  $M$  is isomorphic to a direct summand of a  $c$ -th syzygy of  $N$ .

*Proof.* By virtue of [5, Remark 5.2(1)], there exists an ideal  $I$  of  $R$  with  $\operatorname{NF}(M) = \operatorname{V}(I)$  such that  $I \cdot \operatorname{Ext}_R^i(M, X) = 0$  for all integers  $i > 0$  and all  $R$ -modules  $X$ . As

$$\dim \operatorname{NF}(M) = \dim R/I = d - \operatorname{ht} I,$$

we have  $\operatorname{ht} I = \operatorname{codim} \operatorname{NF}(M) \geq c$ , and can take an  $R$ -sequence  $\mathbf{x} = x_1, \dots, x_c$  in  $I$ . Setting  $N = M/\mathbf{x}M$ , we see from [7, Proposition 2.2] that  $M$  is isomorphic to a direct summand of  $\Omega^c N$ . The condition (3) is thus satisfied, and it is observed that  $N$  is a CM  $R$ -module with  $\operatorname{codim} N = d - \dim N = c$ .

Now it remains to verify that  $N$  satisfies the condition (2). Fix a prime ideal  $\mathfrak{p}$  in the union  $\operatorname{IPD}(N) \cup \operatorname{NF}(M)$ . Then it is easily observed that  $\mathfrak{p}$  contains the sequence  $\mathbf{x}$ . Hence by [3, Exercise 1.3.6] the equalities

$$\operatorname{pd}_{R_{\mathfrak{p}}} N_{\mathfrak{p}} = \operatorname{pd}_{R_{\mathfrak{p}}} M_{\mathfrak{p}}/\mathbf{x}M_{\mathfrak{p}} = \operatorname{pd}_{R_{\mathfrak{p}}} M_{\mathfrak{p}} + c$$

hold. This shows that the  $R_{\mathfrak{p}}$ -module  $N_{\mathfrak{p}}$  has infinite projective dimension if and only if so does  $M_{\mathfrak{p}}$ . Since  $M$  is a MCM  $R$ -module, the Auslander–Buchsbaum formula implies  $\operatorname{IPD}(M) = \operatorname{NF}(M)$ . Therefore we obtain  $\operatorname{IPD}(N) = \operatorname{NF}(M)$ .  $\blacksquare$

As an immediate consequence of the above proposition, the following holds.

**Corollary 2.4.** *Let  $M$  be a MCM  $R$ -module whose nonfree locus has dimension  $n$ . Then there exists a CM  $R$ -module  $N$  of dimension  $n$  such that  $M$  is isomorphic to a direct summand of  $\Omega^{d-n} N$ .*

*Proof.* We have  $\operatorname{codim} \operatorname{NF}(M) = d - n$ . Apply Proposition 2.3 to  $c := d - n$ .  $\blacksquare$

Applying the above corollary to  $n = 0$ , we obtain the following result, which recovers [7, Corollary 2.6].

**Corollary 2.5.** *Let  $M$  be a MCM  $R$ -module which is locally free on the punctured spectrum of  $R$ . Then there exists an  $R$ -module  $N$  of finite length such that  $M$  is isomorphic to a direct summand of  $\Omega^d N$ .*

Next we establish a criterion for  $R$  to satisfy Serre's condition  $(R_n)$  in terms of the codimensions of the nonfree loci of MCM  $R$ -modules.

**Proposition 2.6.** *The following are equivalent for each  $0 \leq c \leq d$ .*

- (1) *The ring  $R$  satisfies  $(R_{c-1})$ .*
- (2) *One has  $\text{codim Sing } R \geq c$ .*
- (3) *One has  $\text{codim NF}(M) \geq c$  for all MCM  $R$ -modules  $M$ .*

*Proof.* (1)  $\Rightarrow$  (2): Let  $\mathfrak{p}$  be a prime ideal in  $\text{Sing } R$ . As  $R$  satisfies  $(R_{c-1})$ , the height of  $\mathfrak{p}$  is at least  $c$ , whence  $\dim R/\mathfrak{p} \leq d - c$ . Therefore  $\dim \text{Sing } R \leq d - c$ , which means that  $\text{Sing } R$  has codimension at least  $c$ .

(2)  $\Rightarrow$  (3): Since  $\text{NF}(M)$  is contained in  $\text{Sing } R$ , we have  $\dim \text{NF}(M) \leq \dim \text{Sing } R$ . Hence the (in)equalities

$$\text{codim NF}(M) = d - \dim \text{NF}(M) \geq d - \dim \text{Sing } R = \text{codim Sing } R \geq c$$

follow.

(3)  $\Rightarrow$  (1): Let  $\mathfrak{p}$  be a prime ideal of  $R$  with  $\text{ht } \mathfrak{p} \leq c-1$ . Let  $M$  be a  $d$ -th syzygy of the  $R$ -module  $R/\mathfrak{p}$ . Then  $M$  is a MCM  $R$ -module, and by assumption we have  $\text{codim NF}(M) \geq c$ , or equivalently,

$$\dim \text{NF}(M) \leq d - c.$$

Suppose that  $R_{\mathfrak{p}}$  is not regular. Then the  $R_{\mathfrak{p}}$ -module  $M_{\mathfrak{p}}$  is not free, for it is a  $d$ -th syzygy of the  $R_{\mathfrak{p}}$ -module  $\kappa(\mathfrak{p})$ . Hence  $\mathfrak{p}$  belongs to  $\text{NF}(M)$ , and there are inequalities

$$\dim \text{NF}(M) \geq \dim R/\mathfrak{p} \geq d - c + 1.$$

This contradiction shows that  $R_{\mathfrak{p}}$  is regular. ■

Let us now state and prove the main result of this section, which characterizes CM local rings satisfying Serre's  $(R_n)$ -condition.

**Theorem 2.7.** *For every integer  $0 \leq c \leq d$  the following are equivalent.*

- (1) *The ring  $R$  satisfies  $(R_{c-1})$ .*
- (2) *Every MCM  $R$ -module is isomorphic to a direct summand of a  $c$ -th syzygy of a CM  $R$ -module of codimension  $c$ .*
- (3) *Every MCM  $R$ -module is isomorphic to a direct summand of some syzygy of an  $R$ -module of codimension at least  $c$ .*

*Proof.* Propositions 2.3 and 2.6 show that (1) implies (2), and it is obvious that (2) implies (3). Assume that (3) holds, and take any MCM  $R$ -module  $M$ . By assumption, there are an  $R$ -module  $N$  with  $\text{codim } N \geq c$  and an integer  $b \geq 0$  such that  $M$  is isomorphic to a direct summand of  $\Omega^b N$ . Then we have inclusions  $\text{NF}(M) \subseteq \text{NF}(\Omega^b N) \subseteq \text{Supp } N$  of closed subsets of  $\text{Spec } R$ , which implies

$$\dim \text{NF}(M) \leq \dim \text{NF}(\Omega^b N) \leq \dim \text{Supp } N = \dim N \leq d - c.$$

Hence  $\text{NF}(M)$  has codimension at least  $c$ , and it is deduced from Proposition 2.6 that  $R$  satisfies  $(R_{c-1})$ . ■

### 3. MCM MODULES THAT ARE DIRECT SUMMANDS OF MCM APPROXIMATIONS

Throughout this section, our ring  $R$  is further assumed to be Gorenstein. The following is a celebrated result due to Auslander and Buchweitz [2, Theorem 1.8].

**Theorem 3.1** (Auslander–Buchweitz). *For each  $R$ -module  $M$  there exists an exact sequence*

$$(3.1.1) \quad 0 \rightarrow Y \rightarrow X \rightarrow M \rightarrow 0$$

*of  $R$ -modules such that  $X$  is MCM and  $Y$  has finite projective dimension.*

**Definition 3.2.** A MCM  $R$ -module  $X$  admitting an exact sequence of the form (3.1.1) is called a *MCM approximation* of  $M$ .

For an  $R$ -module  $M$  we denote by  $\mathrm{Tr}M$  the (*Auslander*) *transpose* of  $M$ , that is, the cokernel of the  $R$ -dual of the first differential map in a free resolution of  $M$ . We denote by  $\underline{\mathrm{MCM}}(R)$  the *stable category of MCM  $R$ -modules*. This is defined as follows: the objects of  $\underline{\mathrm{MCM}}(R)$  are precisely the MCM  $R$ -modules, and the hom-set  $\mathrm{Hom}_{\underline{\mathrm{MCM}}(R)}(M, N)$  of objects  $M, N$  in  $\underline{\mathrm{MCM}}(R)$  is the quotient of  $\mathrm{Hom}_R(M, N)$  by the  $R$ -submodule consisting of homomorphisms factoring through free  $R$ -modules. Since  $R$  is assumed to be Gorenstein,  $\underline{\mathrm{MCM}}(R)$  is a triangulated category, and taking a syzygy and a transpose defines an autoequivalence and a duality of  $\underline{\mathrm{MCM}}(R)$ , respectively.

$$\begin{aligned} \Omega : \underline{\mathrm{MCM}}(R) &\xrightarrow{\cong} \underline{\mathrm{MCM}}(R), \\ \mathrm{Tr} : \underline{\mathrm{MCM}}(R) &\xrightarrow{\cong} \underline{\mathrm{MCM}}(R)^{\mathrm{op}}. \end{aligned}$$

For details, we refer the reader to [1] and [4].

One can describe a MCM approximation by using syzygies and transposes:

**Lemma 3.3.** *For any  $R$ -module  $M$ , the  $R$ -module*

$$\mathrm{Tr}\Omega^n\mathrm{Tr}\Omega^n M$$

*is a MCM approximation of  $M$  for all  $n \geq d - \mathrm{depth} M$ .*

*Proof.* Note that  $\Omega^n M$  is a MCM  $R$ -module. Since both  $\Omega$  and  $\mathrm{Tr}$  preserve the MCM property, the  $R$ -module  $X = \mathrm{Tr}\Omega^n\mathrm{Tr}(\Omega^n M)$  is also a MCM module. It follows from [1, Proposition (2.21) and Corollary (4.22)] that there exists an exact sequence

$$(3.3.1) \quad 0 \rightarrow K \rightarrow X \rightarrow M \rightarrow 0$$

of  $R$ -modules such that  $K$  has projective dimension at most  $n - 1$ . Consequently,  $X$  is a MCM approximation of  $M$ . ■

A MCM approximation version of Corollary 2.4 also holds true:

**Proposition 3.4.** (1) *Let  $M$  be a MCM  $R$ -module with  $n$ -dimensional nonfree locus. Then there exists an  $n$ -dimensional CM  $R$ -module  $N$  such that  $M$  is isomorphic to a direct summand of a MCM approximation of  $N$ .*  
 (2) *Let  $M$  be a MCM  $R$ -module that is locally free on the punctured spectrum of  $R$ . Then there exists an  $R$ -module  $N$  of finite length such that  $M$  is isomorphic to a direct summand of a MCM approximation of  $N$ .*

*Proof.* (1) It is easy to see that  $\mathrm{NF}(\Omega^{d-n}M)$  coincides with  $\mathrm{NF}(M)$ . Applying Corollary 2.4 to the MCM module  $\Omega^{d-n}M$ , we find a CM module  $N$  of dimension  $n$  such that

$\Omega^{d-n}M$  is isomorphic to a direct summand of  $\Omega^{d-n}N$ . Taking  $\mathrm{Tr}\Omega^{d-n}\mathrm{Tr}$  yields that  $M$  is isomorphic to a direct summand of

$$X := \mathrm{Tr}\Omega^{d-n}\mathrm{Tr}\Omega^{d-n}N \oplus F$$

for some free  $R$ -module  $F$ . Using Lemma 3.3, we easily see that  $X$  is a MCM approximation of  $N$ .

(2) The assertion follows from applying (1) to  $n = 0$ . ■

The main result of this section is the following characterization of Gorenstein local rings satisfying Serre's condition  $(R_n)$ . This result can be viewed as a MCM approximation version of Theorem 2.7.

**Theorem 3.5.** *The following are equivalent for each  $0 \leq c \leq d$ .*

- (1)  *$R$  satisfies  $(R_{c-1})$ .*
- (2) *Every MCM  $R$ -module is isomorphic to a direct summand of a MCM approximation of a CM  $R$ -module of codimension  $c$ .*
- (3) *Every MCM  $R$ -module is isomorphic to a direct summand of a MCM approximation of an  $R$ -module of codimension at least  $c$ .*

*Proof.* (1)  $\Rightarrow$  (2): Let  $M$  be a MCM  $R$ -module. Using Theorem 2.7 for the MCM  $R$ -module  $\Omega^c M$ , we get a CM  $R$ -module  $N$  of codimension  $c$  such that  $\Omega^c M$  is isomorphic to a direct summand of  $\Omega^c N$ . Then applying  $\mathrm{Tr}\Omega^c\mathrm{Tr}$  to this relation shows that  $\mathrm{Tr}\Omega^c\mathrm{Tr}\Omega^c M$  is isomorphic to a direct summand of  $X := \mathrm{Tr}\Omega^c\mathrm{Tr}\Omega^c N$  up to free summands. By Lemma 3.3 the module  $X$  is a MCM approximation of  $N$ . Since we have a duality

$$\mathrm{Tr}\Omega^c : \underline{\mathrm{MCM}}(R) \xrightarrow{\cong} \underline{\mathrm{MCM}}(R),$$

the  $R$ -module  $\mathrm{Tr}\Omega^c\mathrm{Tr}\Omega^c M$  is isomorphic to  $M$  up to free summands. Therefore  $M$  is isomorphic to a direct summand of  $X \oplus F$  for some free  $R$ -module  $F$ . It is easy to see that  $X \oplus F$  is also a MCM approximation of  $N$ .

(2)  $\Rightarrow$  (3): This implication is obvious.

(3)  $\Rightarrow$  (1): Let  $M$  be a MCM  $R$ -module. Then  $N := \mathrm{Tr}\Omega^d\mathrm{Tr}M$  is also a MCM  $R$ -module. Applying the condition (3) to  $N$ , we observe that there exists an  $R$ -module  $L$  of codimension at least  $c$  such that  $N$  is isomorphic to a direct summand of a MCM approximation  $X$  of  $L$ . It follows from [2, Theorem B] and Lemma 3.3 that the  $R$ -module  $X$  is isomorphic to  $\mathrm{Tr}\Omega^d\mathrm{Tr}\Omega^d L$  up to free summands. The functor

$$\mathrm{Tr}\Omega^d\mathrm{Tr} : \underline{\mathrm{MCM}}(R) \xrightarrow{\cong} \underline{\mathrm{MCM}}(R)$$

is an equivalence, so we see that  $M$  is isomorphic to a direct summand of  $\Omega^d L$  up to free summands. Thus Theorem 2.7 implies that  $R$  satisfies Serre's condition  $(R_{c-1})$ . ■

*Proof of Theorem 1.2.* The assertion follows by combining Theorems 2.7 and 3.5. ■

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## REFERENCES

- [1] M. AUSLANDER; M. BRIDGER, Stable module theory, *Mem. Amer. Math. Soc.* No. 94, *American Mathematical Society, Providence, R.I.*, 1969.
- [2] M. AUSLANDER; R.-O. BUCHWEITZ, The homological theory of maximal Cohen–Macaulay approximations, Colloque en l'honneur de Pierre Samuel (Orsay, 1987), *Mém. Soc. Math. France (N.S.)* No. 38 (1989), 5–37.
- [3] W. BRUNS; J. HERZOG, Cohen–Macaulay rings, revised edition, Cambridge Studies in Advanced Mathematics, 39, *Cambridge University Press, Cambridge*, 1998.
- [4] R.-O. BUCHWEITZ, Maximal Cohen–Macaulay modules and Tate-cohomology over Gorenstein rings, Preprint (1986), <http://hdl.handle.net/1807/16682>.
- [5] H. DAO; R. TAKAHASHI, The dimension of a subcategory of modules, Preprint (2012), [arXiv:1203.1955v2](https://arxiv.org/abs/1203.1955v2).
- [6] K. KATO, Syzygies of modules with positive codimension, *J. Algebra* **318** (2007), no. 1, 25–36.
- [7] R. TAKAHASHI, Classifying thick subcategories of the stable category of Cohen–Macaulay modules, *Adv. Math.* **225** (2010), no. 4, 2076–2116.

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